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# Thermal Stresses in Nonhomogeneous Thin Shells

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**Thermal stresses in nonhomogeneous, complete, thin shells of arbitrary shape and constant thickness are evaluated for the case where the temperature and the elastic constants are a function of only the local normal coordinate. The analysis shows that the thermal stresses in the local tangent plane are independent of the shape of the middle surface of the shell.**

## Introduction

The design of high-temperature thermal protection systems for re-entry vehicles, rocket nozzles, and hypervelocity aircraft requires consideration of multilayered shells with large temperature variations through the shell thickness. The problem of thermal stresses in such structures has received considerable attention in recent years. The solutions for the most arbitrary geometry and temperature distributions are cumbersome and require considerable use of high-speed computing equipment. Such solutions are of limited value for the parametric studies necessary to design minimum weight vehicles, since they must be used in conjunction with equally complex aerothermodynamic analyses. The aim of this paper is to present the simple result of an analysis that treats a major aspect of these thermal stress problems in a fashion that makes it readily usable for the multiple analyses associated with system optimization.

## Analysis

The problem to be treated is an arbitrary shape, closed, thin shell of constant thickness subjected to an arbitrary temperature distribution that varies through the thickness only. Mechanical properties of the shell may vary with respect to the local normal coordinate and also with respect to temperature. The material is treated as isotropic and elastic. The surface coordinate curves are selected to coincide with the lines of principal curvature of the surface. The local normal is taken as the direction of the third coordinate of this orthogonal coordinate system.

The radial stresses are neglected in the tangential stress-strain relations, which may be written as

$$\begin{aligned}\sigma_{11} &= [E/(1-\nu^2)][\epsilon_{11}+\nu\epsilon_{22}-(1+\nu)\epsilon_T]\\ \sigma_{22} &= [E/(1-\nu^2)][\epsilon_{22}+\nu\epsilon_{11}-(1+\nu)\epsilon_T]\\ \sigma_{12} &= [E/2(1+\nu)]\epsilon_{12}\end{aligned} \tag{1}$$

where $E$ is Young's modulus, $\nu$ is Poisson's ratio; 1 and 2 are the principal directions, and $\epsilon_T$ is the free thermal expansion associated with the prescribed temperature changes.

The equilibrium equations for an arbitrary thin shell are given by Novozhilov[1] as

$$\frac{\partial(A_2T_1)}{\partial\alpha_1}+\frac{\partial(A_1S)}{\partial\alpha_2}+S\frac{\partial A_1}{\partial\alpha_2}-T_2\frac{\partial A_2}{\partial\alpha_1}+\frac{1}{R_1}\left[\frac{\partial(A_2M_1)}{\partial\alpha_1}-M_2\frac{\partial A_2}{\partial\alpha_1}+2\frac{\partial(A_1H)}{\partial\alpha_2}+2H\frac{R_1}{R_2}\frac{\partial A_1}{\partial\alpha_2}\right]=-A_1A_2q_1$$

$$\frac{\partial(A_2S)}{\partial\alpha_1}+\frac{\partial(A_1T_2)}{\partial\alpha_2}+S\frac{\partial A_2}{\partial\alpha_1}-T_1\frac{\partial A_1}{\partial\alpha_2}+\frac{1}{R_2}\left[\frac{\partial(A_1M_2)}{\partial\alpha_2}-M_1\frac{\partial A_1}{\partial\alpha_2}+2\frac{\partial(A_2H)}{\partial\alpha_1}+2H\frac{R_2}{R_1}\frac{\partial A_2}{\partial\alpha_1}\right]=-A_1A_2q$$

$$\frac{T_1}{R_1}+\frac{T_2}{R_2}-\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left[\frac{\partial(A_2M_1)}{\partial\alpha_1}+\frac{\partial(A_1H)}{\partial\alpha_2}+H\frac{\partial A_1}{\partial\alpha_2}-M_2\frac{\partial A_2}{\partial\alpha_1}\right]+\frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left[\frac{\partial(A_2H)}{\partial\alpha_1}+\frac{\partial(A_1M_2)}{\partial\alpha_2}+H\frac{\partial A_2}{\partial\alpha_1}-M_1\frac{\partial A_1}{\partial\alpha_2}\right]\right\}=q_n \tag{2}$$

where $\alpha_1$ and $\alpha_2$ are the surface coordinates, $R_1$ and $R_2$ are the principal radii of curvature, $q_1$, $q_2$, and $q_n$ are the components of the applied pressure, $A_1$ and $A_2$ are Lamés parameters, defined by the metric form

$$(ds)^2=(A_1d\alpha_1)^2+(A_2d\alpha_2)^2$$

and $T_1$, $T_2$, $T_{12}$, and $T_{21}$ are stress resultants, and $M_1$, $M_2$, $M_{12}$, and $M_{21}$ are moment resultants defined by (for shell thickness $h$)

$$\begin{aligned}T_1 &= \int_{-h/2}^{h/2}\sigma_{11}dz & T_2 &= \int_{-h/2}^{h/2}\sigma_{22}dz\\ T_{12} &= \int_{-h/2}^{h/2}\sigma_{12}\left(1+\frac{z}{R_2}\right)dz & T_{21} &= \int_{-h/2}^{h/2}\sigma_{21}\times\left(1+\frac{z}{R_1}\right)dz\\ M_1 &= \int_{-h/2}^{h/2}\sigma_{11}zdz & M_2 &= \int_{-h/2}^{h/2}\sigma_{22}zdz\\ M_{12} &= \int_{-h/2}^{h/2}\sigma_{12}\left(1+\frac{z}{R_2}\right)zdz & M_{21} &= \int_{-h/2}^{h/2}\sigma_{21}\times\left(1+\frac{z}{R_1}\right)zdz\end{aligned} \tag{3}$$

$$\begin{aligned}H &= \tfrac{1}{2}(M_{12}+M_{21})\\ S &= T_{12}-(M_{21}/R_2)=T_{21}-(M_{12}/R_1)\end{aligned}$$

Note that, although $z/R \ll 1$, the simplification that results from neglecting these terms in the shear force and twisting moment resultants introduces certain contradictions into the theory.[1] For this reason, such terms will be retained here, and the resultants are as defined in Eqs. (3).

It can be seen that, for no external load ($q_1 = q_2 = q_n = 0$), compatible displacements which yield equal, constant moment resultants in the two principal directions, and vanishing stress and twisting moment resultants, will satisfy the equilibrium equations (2) and thus be a solution to the problem. (Since a closed shell is considered, there are no further boundary conditions.) Such a solution is given by the uniform strain solution:

$$\epsilon_{11}=\epsilon_{22}=\frac{\displaystyle\int_{-h/2}^{h/2}\frac{E\epsilon_T}{1-\nu}dz}{\displaystyle\int_{-h/2}^{h/2}\frac{E}{1-\nu}dz} \tag{4}$$

$$\epsilon_{12}=0$$

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Substitution of Eqs. (4) into Eqs. (1) yields

$$\sigma_{11} = \sigma_{22} = \frac{E}{1-\nu}\left[\frac{\int_{-h/2}^{h/2} \frac{E\epsilon_T}{1-\nu}\,dz}{\int_{-h/2}^{h/2} \frac{E}{1-\nu}\,dz} - \epsilon_T\right] \qquad (5)$$

$$\sigma_{12} = 0$$

Equations (5) and (3) define the resultants:

$$T_1 = T_2 = T_{12} = M_{12} = M_{21} = H = S = 0$$
$$M_1 = M_2 \qquad (6)$$

The resultants (6) satisfy Eqs. (2), and hence the stresses of Eqs. (5) are the desired solution.

This analysis of closed, thin shells of constant thickness, subjected to temperature variations through the thickness, shows that, within the limits of thin-shell theory, the tangential thermal stresses are independent of the shape of the shell. The radial stresses depend upon the local curvatures and hence vary over the shell. The expressions for thermal stresses, Eqs. (5), include the effects of layers of different materials through the shell thickness as well as variations of mechanical properties with temperature.

### Reference

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# Modification of Encke's Method Suitable for Analog Solution

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### Nomenclature

- $\mathbf{e}_h$ = a unit vector in the plane formed by $\mathbf{r}_s$ and $\dot{\mathbf{r}}_s$ and perpendicular to $\mathbf{r}_s$
- $\mathbf{F}$ = force vector
- $g_1, g_2$ = relative gravitational components defined by Eqs. (6) and (7)
- $\delta h$ = difference between $|\mathbf{r}|$ and $|\mathbf{r}_s|$
- $m$ = particle mass
- $\mathbf{r}$ = particle position vector
- $\mathbf{r}_s$ = reference particle position vector
- $\mathbf{S}_s$ = relative range vector
- $S_x$ = inertial component of $\mathbf{S}_s$ aligned perpendicular to $\mathbf{r}$ at zero time and in the direction of motion
- $S_y$ = inertial component of $\mathbf{S}_s$, perpendicular to the reference plane, and completing an orthogonal system with $S_x$ and $S_z$
- $S_z$ = inertial component of $\mathbf{S}_s$ aligned along $-\mathbf{r}$ at zero time
- $\mu$ = planetary gravitational constant
- $\delta\rho$ = nondimensional altitude difference defined by Eq. (8)
- $\chi$ = angle between $\mathbf{r}$ and $\mathbf{r}_s$
- $\cdot$ = first derivative with respect to time
- $\cdot\cdot$ = second derivative with respect to time

THE method proposed by J. F. Encke for handling small perturbations in the vicinity of a central force field is peculiarly suitable for real-time solution of the orbital flight equations on an analog computer. The method requires that differential accelerations due to perturbations from a two-body reference orbit be integrated; thus, the kinematic solution for position becomes the relative position vector. The number of significant figures needed in the calculation is reduced to a level where computation with four digits becomes comparable to a six-digit solution achieved by integrating the total accelerations.

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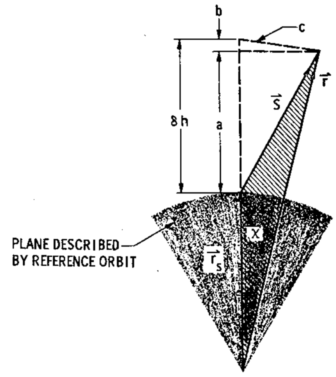


**Fig. 1 Relative orientation of r and $\mathbf{r}_s$ vectors**

The vector differential equation of motion in a central force field is given as

$$\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r} + (\mathbf{F}/m) \qquad (1)$$

Now proceed to derive the differential equation of motion of the relative range vector $\mathbf{S}$ referenced to a point particle operating in freefall at a position $\mathbf{r}_s$ such that

$$\mathbf{r} = \mathbf{r}_s + \mathbf{S}_s \qquad (2)$$

From this it follows that $\dot{\mathbf{r}} = {}^{\cdot}{}_s + \dot{\mathbf{S}}_s$. Substitution of (2) into (1) yields

$$\ddot{\mathbf{S}}_s = -\ddot{\mathbf{r}}_s - (\mu/r^3)\mathbf{r}_s - (\mu/r^3)\mathbf{S}_s + (\mathbf{F}/m) \qquad (3)$$

For the conditions stated concerning $\mathbf{r}_s$,

$$\ddot{\mathbf{r}}_s = -(\mu/r_s^3)\mathbf{r}_s \qquad (4)$$

Substitution of (4) into (3) yields

$$\ddot{\mathbf{S}}_s = -\mathbf{r}_s \frac{\mu}{r_s^3}\left(\frac{r_s^3}{r^3} - 1\right) - \frac{\mu}{r_s^3}\frac{r_s^3}{r^3}\mathbf{S}_s + \frac{\mathbf{F}}{m} \qquad (5)$$

Now define

$$g_1 \equiv (\mu/r_s^3)[(r_s^3/r^3) - 1] = -g_2\delta\rho[2 + \delta\rho + (1+\delta\rho)^2] \qquad (6)$$

$$g_2 \equiv (\mu/r_s^3)(r_s^3/r^3) = (\mu/r_s^3)[1/(1+\delta\rho)^3] \qquad (7)$$

$$\delta\rho \equiv (r/r_s) - 1 \qquad (8)$$

Then the resulting relation describing $\mathbf{S}$ is obtained by twice integrating

$$\ddot{\mathbf{S}}_s = -\mathbf{r}_s g_1 - \mathbf{S}_s g_2 + (\mathbf{F}/m) \qquad (9)$$

It is evident that the subtraction described by the identity of Eq. (8) does not lead to the desired result of fewer significant

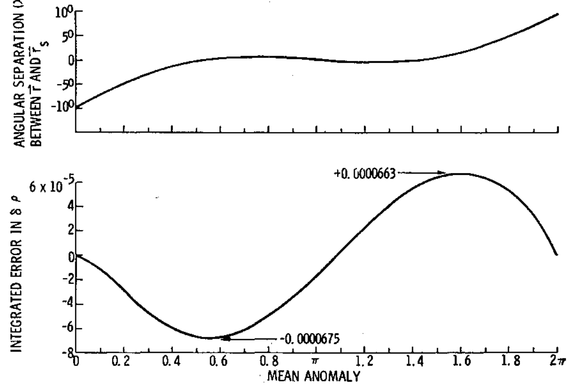


**Fig. 2 Integrated error in $\delta\rho$ due to small angle approximation made for $\chi$**